Intrinsic Local Distances

A Mixed Solution to Weyl's Tile Argument

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Paradoxes of Continuous Spacetime

Measure-Theoretic Paradoxes

– *Zeno's Paradox of Measure*. Zeros add up to one?

Paradoxes of Continuous Spacetime

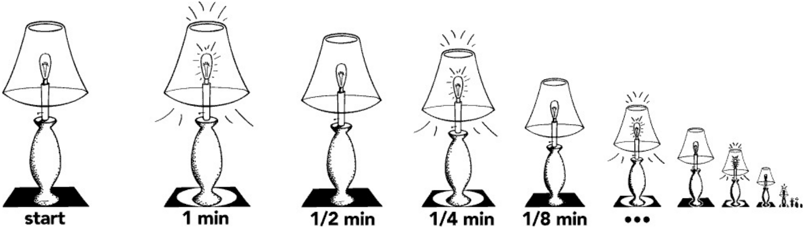
Measure-Theoretic Paradoxes

- *Zeno's Paradox of Measure*. Zeros add up to one?
- *Banach-Tarski Paradox*. One sphere becomes more. (No stretching!)

Paradoxes of Continuous Spacetime

Infinite Divisibility

– *Thomson's Lamp*. Turn on and off alternatively.



Paradoxes of Continuous Spacetime

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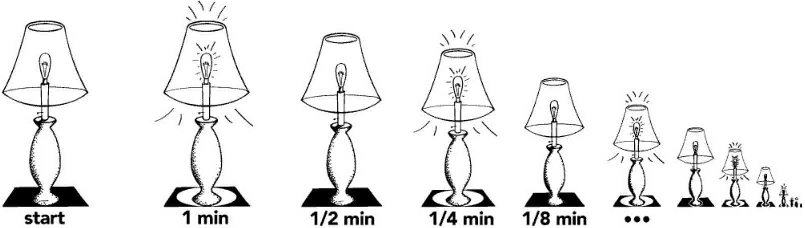


– *Faris' Sheet*. Write "1" $\longrightarrow$ erase $\longrightarrow$ write "2" $\longrightarrow$ erase...

Paradoxes of Continuous Spacetime

Cracks in Physics [Baez 2018]

The trouble with point particles

[Grøn 2011]

A LONELY PARTICLE ACCELERATES ITSELF TO THE SPEED OF LIGHT

Paradoxes of Continuous Spacetime

Cracks in Physics [Baez 2018]

The trouble with point particles

[Grøn 2011]

A lonely particle accelerates itself to the speed of light

[Eliezer 1943]

Opposite-charged particles run away from each other

So...?

Continuous spacetime is bizarre!

Discrete Spacetime

Atomism. Spacetime is composed of (finitely) extended *"atoms."*

Atom = indivisible part of spacetime

Weyl's Tile Argument

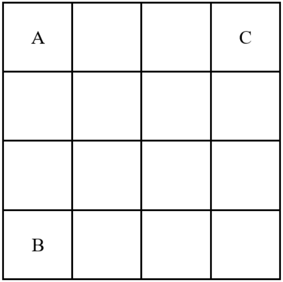


Figure 1

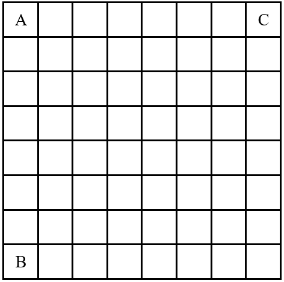


Figure 2

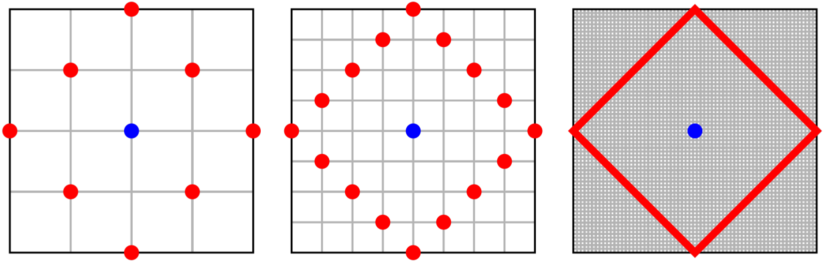
Weyl's tile argument

If atomism is true, then the Pythagorean theorem is not approximately true.

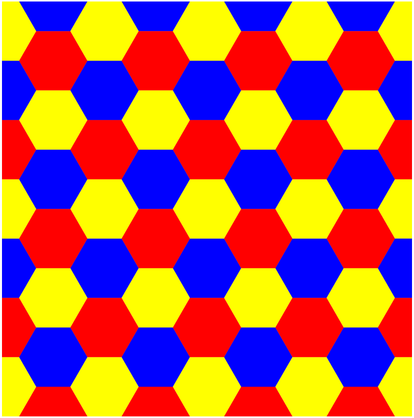
Weyl's Tile Argument

Distance-by-Counting. The distance between two atoms is the least number of atoms connecting one to the other. [Riemann 1866]

Taxicab Geometry

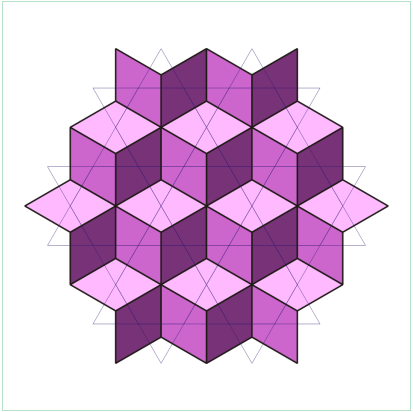


Different tiling models?



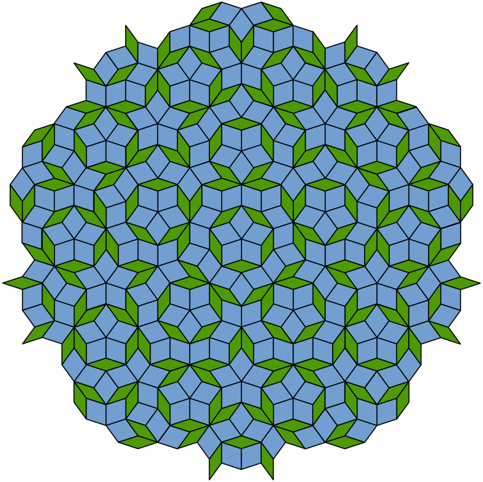
Hexagonal tiling?

Different tiling models?



Rhombille tiling?

Different tiling models?



Penrose tiling?

I am going to replace this...

Distance-by-Counting. The distance between two atoms is the least number of atoms connecting one to the other.

Defining the Success Conditions

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- Compatible with physics as we know it
- Intelligible, not counterintuitive, natural, motivated...

...by Intrinsic Distances?

Intrinsic Account of Distance

The distance between any two atoms is intrinsic to them.

Let an atom be represented by a pair of integers.

Euclidean Model [McDaniel 2007]

The distance between atoms $(a_1, b_1), (a_2, b_2)$ is $\sqrt{(a_2 - a_1)^2 + (b_2 - b_1)^2}$.

However...

General Relativity [Einstein 1916]

The metric of spacetime is related definitely to the distribution of mass-energy by Einstein field equations.

$\Rightarrow$ A massive body between two faraway atoms will distort their distance.

Evaluating Intrinsic Distances

✓ Allows atomistic space to approximate Euclidean geometry

✓ Intelligible, not counterintuitive, natural, motivated...

✗ Compatible with physics as we know it

The Mixed Account

Intrinsic Local Distances

Some atoms bear *primitive distances*.

Path-dependent Distances

Distance between two atoms = the least sum of primitive distances from one to the other.

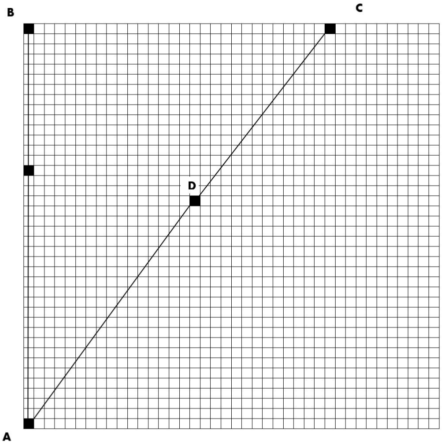
The Mixed Account

Atoms are still represented by pairs of integers.

Euclidean Model in The Mixed Account

For any atoms a, b, their primitive distance is $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ iff $(x_2 - x_1)^2 + (y_2 - y_1)^2 \leq \Delta^2$.

Δ is a large number.



$$\Delta = 30$$

$$A : (0, 0), B : (0, 40)$$

$$C : (30, 40), D : (15, 21)$$

$$AB = 39$$

$$BC = 32$$

$$AB^2 + BC^2 = 50^2$$

$$AD = 25.81$$

$$DC = 24.21$$

$$AC = AD + DC = 50.02$$

Euclidean Approximation

A metric space X with a metric d is ϵ-*isometric* to Euclidean space E with regard to r iff there is a map f from X to E such that (1) for $x, y \in X$, we have

$$1 - \epsilon \leq \frac{e(f(x), f(y))}{d(x, y)} \leq 1 + \epsilon$$

(2) for every $p \in E$, there is a $x \in X$ such that $e(p, f(x)) \leq r$.

Approximation Theorem

For any ϵ and r, there is a set of points with a shortest path metric (with distances being bounded by a finite number) that is ϵ-isometric to Euclidean space with regard to r.

Application

Suppose the shortest distance = the Planck length ($10^{-35}m$)

Suppose the relative accuracy of our measurement = $\pm 1.6 \times 10^{-9}$

The longest primitive distance = the diameter of a neutrino ($10^{-26}m$)

Evaluating The Mixed Account

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  - ☑ Primitive distance
  - ☑ Path-dependent distance
  - ☐ Primitive + path-dependent

Continuous Space (Manifold)

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The length of a path is equal to the path integral of the lengths of the tangent vectors along the path.

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The length of a path is equal to the path integral of the lengths of the tangent vectors along the path.

Length of tangent vectors $\approx$ primitive distances

path integral $\approx$ sum of primitive distances

☑ Primitive + path-dependent

So...

The Mixed Account successfully solves Weyl's tile argument.

Forrest's Solution

Forrest endorses **Distance-by-counting**.

Forrest's Model [Forrest 1995]

Two atoms (x_1, y_1) and (x_2, y_2) are *adjacent/connected* iff $(x_2 - x_1)^2 + (y_2 - y_1)^2 \leq m^2$.

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- adjacent atoms $\approx$ neighbors
- All neighbors bear distance 1.

Forrest's Solution

- ✓ Allows atomistic space to approximate Euclidean geometry
- ✓ Compatible with physics as we know it
- ✓ Intelligible, not counterintuitive, natural...

So, Forrest's solution is also successful. (But...)

But...

Forrest's model is not locally approximately Euclidean.

Disadvantages:

- Perhaps our space is locally approximately Euclidean
- Redundant structure that plays no role in physical theories
- Possibly unmotivated: the atoms are much smaller than the Planck level