Why the Weyl Tile Argument is Wrong

Lu Chen

Koc University (Istanbul, Turkey)

Why Discrete Spacetime

Discrete space(time). Space(time) is composed of indivisible parts of a finite extension.

The appeal of discrete spacetime:

- conceptual difficulties with continuum
- lasting difficulty with classical physics
- difficulties with quantum gravity

Classical Physics

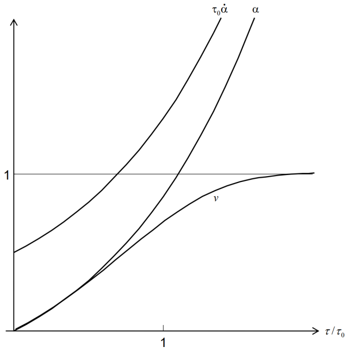


Figure: Runaway particle accelerating to the speed of light (Grøn 2011)

Quantum gravity

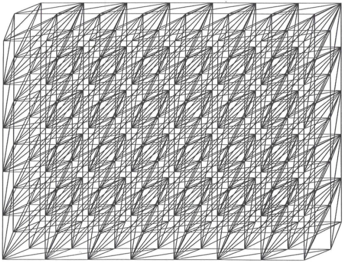


Figure: Spacetime lattice composed of 4-simplices (Hamber 2008)

Weyl's Tile Argument

P1. If space is discrete, then the Pythagorean theorem is not even approximately true (at ordinary scales).
P2. The Pythagorean theorem is approximately true (at ordinary scales).
C. Space is not discrete.

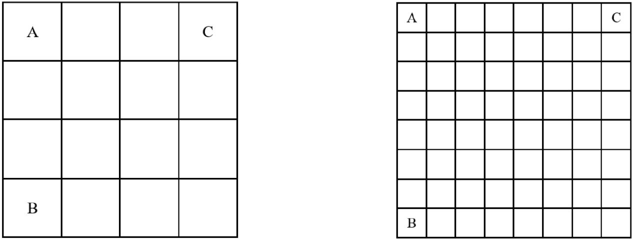


Figure: (left) 4×4 tile space; (right) 8×8 tile space (Salmon 1980)

Weyl's Tile Argument

Implicit Assumption

Distance-by-Counting. The distance between two tiles is the least number of tiles connecting one to the other. [Riemann 1866]

Primitive structure: sets of "tiles", how they connect (topology).

Current proposals

- Adhere to distance-by-counting (Forrest 1995)
- Counting is not enough (van Bendegem 1987, 1997, Chen 2020)

Backstory...

Backstory

Phil introduced the argument to me.

- Geometry vs Dynamics
- Dissertation stage
- Afterwards

My Thesis: Weyl's argument is wrong

The argument misses a realistic possibility that large-scale distance or empirical geometry is not directly determined by fundamental spacetime structure.

GEOMETRICISM. There is a fundamental metric structure that determines empirical geometry (e.g., measured by rigid rods and light rays) independently of dynamical laws.

My thesis: against geometricism

Against geometricism

Empirical geometry can be determined by dynamical laws that do not presuppose a fundamental geometry.

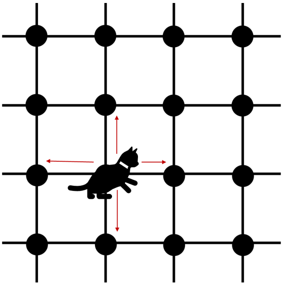
I will demonstrate through two cases:

- Case 1: Random Walk
- Case 2: Quantum mechanics (discrete)

(The following content will be mathematical.)

Case 1: Random Walk

The random walk case that I will focus on describes a tiny cat walking on a square lattice randomly.



Primitives
Space:
lattice points
connectedness
Time:
discrete steps

Figure: A random walk on a square lattice

- at any time step, the cat is at a lattice point.
- at the next step, the cat moves to one of the neighboring points at a $1/4$ chance.

Case 1: Random Walk

Assume that the only important observable is the chance of the cat showing up at a lattice point at a time.

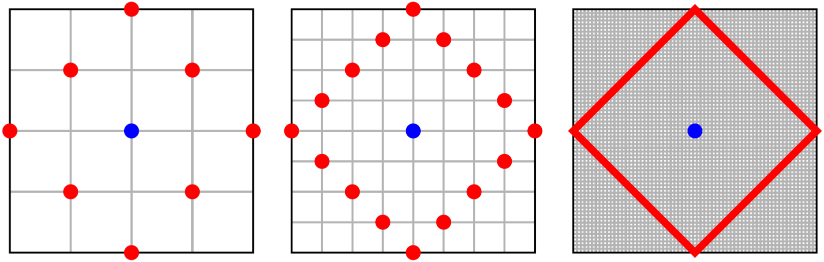
Question

Given an origin, what is the probability distribution like after some time?

The Euclidean case

The distribution is isotropic because Euclidean space is isotropic (no direction is privileged over any other).

Classical motion



Case 1: Random Walk

Theorem 1

ISOTROPY For any starting point $x \in \mathbb{Z}^2$, and for any two points that have approximately the same Euclidean distance to x, the probabilities of the cat showing up in them (if nonzero) are approximately the same after a sufficiently long time.

Note: "Euclidean distance" in ISOTROPY is not a primitive geometrical structure, but merely a feature of our reference (coordinate system).

Random Walk

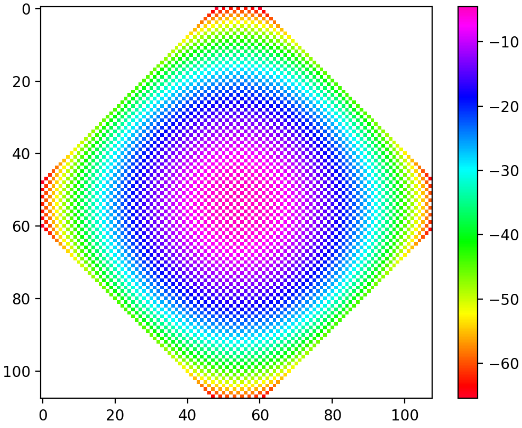


Figure: the probability distribution at $t = 60$ on the square lattice.

The proof (main results)

The probability of reaching (x, y) from $(0, 0)$ at time $= t$ is zero if $x + y$ is of the same parity as t and otherwise equal to:

$$(1/4)^t \binom{t}{(t+x+y)/2} \binom{t}{(t+x-y)/2} \tag{1}$$

which is approximately equal to the following when $x \ll t, y \ll t$ (based on Gallager 1968):

$$\frac{2e^{-(x^2+y^2)/t}}{\pi\sqrt{t^2-(x^2+y^2)}} \tag{2}$$

Morals

- It follows from Isotropy that there is an embedding from discrete space into Euclidean space that approximately preserves the probability distribution of the cat after sufficiently long time.
- In this sense, Euclidean geometry "emerges" from the tile space under the dynamics of a random walk.

Random Walk

- **Advantages.** The case is conceptually clear and rigorous: primitive notions, dynamical laws, mathematical proof, etc.
- **Too hypothetical?** Random walk is used in statistical physics. Nothing like this in fundamental physics?
- To give a more realistic example, I turn to quantum mechanics, in which isotropy also holds (to a degree).

Case 2: Quantum Mechanics: a negative claim

Assume that the only observables are the amplitudes of quantum wavefunctions on lattice points.

Theorem 2

ANISOTROPY. For any quantum mechanical system with initial position $x \in \mathbb{Z}^2$, its wavefunction will *not* evolve to be approximately isotropic for any significant period of time.

A negative claim

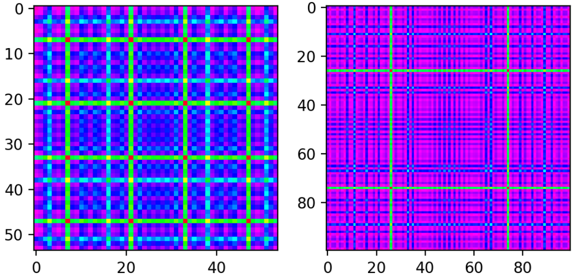


Figure: The time evolution of a quantum mechanical system with its initial position at a single lattice point. (Left) $t = 30$; (Right) $t = 100$.

As before, the scale of each plot is adjusted so that we focus on the significant part of the wavefunction. The plaid shirt pattern remains for later times, with only the colors and frequencies of the stripes changing.

Case 2: Quantum Mechanics

Theorem 3

ISOTROPY-QM. For any quantum mechanical system with its initial position spread out in a sufficiently large region $A \subset \mathbb{Z}^n$, its time evolution is approximately isotropic (i.e., its unitary operator commutes with rotation).

A special case

If the starting position A is spread out and isotropic, then its evolution will continue to be approximately isotropic.

Quantum mechanics

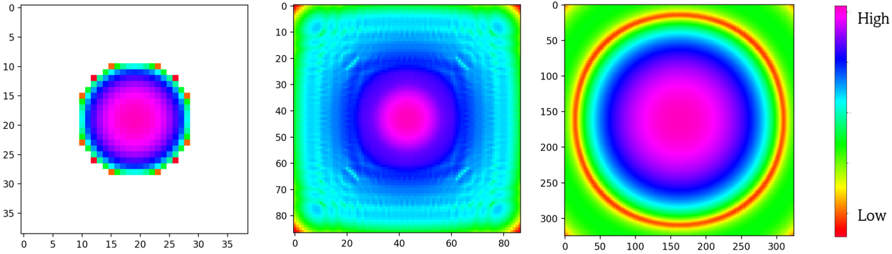


Figure: The evolution of a wavefunction with initial radius of 10. All significant parts of the wavefunction are plotted. (Left) $t = 0$; (Middle) $t = 30$; (Right) $t = 300$.

Note that the scales are different between plots, since the wavefunction is more spread out as time passes; also, the values represented by the same colors are different between plots, since the amplitudes generally get much lower as the wavefunction is spread thin.

The proof (selective steps)

First, discretize the Schrödinger equation. Standard version (setting the Planck constant $\hbar$ to one):

$$i\frac{d}{dt}\Psi(t) = \hat{H}\Psi(t), \tag{3}$$

Discrete position wavefunction:

$$\Psi(t) : \mathbb{Z}^n \to \mathbb{C}.$$

Discrete Hamiltonian:

$$\hat{H}\Psi(t,x) = -\frac{1}{2}\sum_i (\Psi(t, x + e_i) + \Psi(t, x - e_i) - 2\Psi(t, x)) \tag{4}$$

The proof (selective steps)

Discrete Fourier transformation between position space and momentum space:

- Position space → momentum space:

$$\tilde{\Psi}(t,p) = \sum_{x\in\mathbb{Z}^n} e^{-2\pi ipx}\Psi(t,x) \tag{5}$$

- Momentum space → position space:

$$\Psi(t,x) = \int_{p\in B} e^{2\pi ipx}\tilde{\Psi}(t,p)dp \tag{6}$$

The proof (selective steps)

Let $\textbf{Cow}(p) = -\frac{1}{2}\sum_i (e^{2\pi i p_i} + e^{-2\pi i p_i} - 2)$. We can solve the Schrödinger equation in the momentum space:

$$\tilde{\Psi}(t,p) = e^{-i\tilde{H}t}\tilde{\Psi}(0,p) = e^{-i\textbf{Cow}(p)t}\tilde{\Psi}(0,p) \tag{7}$$

$\textbf{Cow}(p)$ is approximately rotationally invariant when p is sufficiently small:

$$\textbf{Cow}(p) = 2\pi^2 \sum_i p_i^2 - O(p^4) \tag{8}$$

Therefore:

$$\tilde{\Psi}(t,Ap) = e^{-i\textbf{Cow}(Ap)t}\tilde{\Psi}(0,Ap) \approx e^{-i\textbf{Cow}(p)t}\tilde{\Psi}(0,Ap) = e^{-i\tilde{H}t}\tilde{\Psi}(0,Ap)$$

Conclusion: Why the Weyl Tile argument is wrong

Geometricist Assumption

Distances and in general the physical geometry can be determined independently from the dynamical laws, and a fundamental geometry must be presumed by dynamics.

Dynamics plays a crucial role in determining physical geometry:

- The observable isotropy is determined by the dynamical laws that govern the movement of the tiny cat in the random walk case and the evolution of the wavefunction in the case of quantum mechanics.
- The discrete version of the Schrödinger equation only requires the topological structure and the derived "differential" structure.

How far can we go?

Strictly speaking, we haven't got rid of all fundamental spacetime structure: the topological structure or just the bare set of lattice points.

- Here, I only focus on the fundamental metric structure.
- Getting rid of all spacetime structure needs a different framework, beyond the current scope.
- I believe it's possible (cf Chen and Fritz 2021).

–The End–